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On Positive Functions in Haar Spaces

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Let X be a compact topological space and $C(X)$ the space of real valued continuous functions on X. For $f \in C(X)$ define

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

Let $\phi_1, \phi_2, \dots, \phi_n$ be a linearly independent subset of $C(X)$. The problem of one sided Chebyshev approximation, which has been of interest to Dunham [2] and others, is the following. Given an $f \in C(X)$, minimize

$$\left\| f - \sum_{i=1}^{n} a_i \phi_i \right\|$$

over all $\sum_{i=1}^{n} a_i \phi_i(x)$ subject to

$$\sum_{i=1}^{n} a_i \phi_i(x) - f(x) \geqslant 0 \qquad \forall x \in X.$$

In this paper we establish the existence of a positive function in a class of Haar spaces which is helpful in proofs of existence theorems. Let H_n denote a Haar subspace of $C(X)$ of dimension n, with basis $\phi_1, \phi_2, \dots, \phi_n$.

In [2], Dunham presented a theorem on existence and uniqueness of a best one-sided approximation to a given $f \in C(X)$, where X is a compact normal space. An essential part of his proof involved the existence of a positive function, and he claimed that if $\phi_1, \dots, \phi_n$ is a Chebyshev set then existence of a positive linear combination $\sum_{i=1}^{n} a_i \phi_i$ was assured. This, however, is not the case, as we see from the following example. Let $X = [-2, -1] \cup [1, 2]$ and let $\phi_1(x) = x$. Then $\phi_1(x)$ has no zeros in X but $\alpha\phi_1(x)$ is positive for no α. We now present a theorem establishing the existence of a positive linear combination under certain conditions.

THEOREM 1. *Let H_n be an n-dimensional Haar subspace of $C(X)$, where X is a compact metric space with a distance function d. If X contains a point p*

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such that p is not an isolated point of X and $X \setminus \{p\}$ is arcwise connected, then there exists $g \in H_n$ such that $g(x) > 0$ for each $x \in X$.

Proof. We will first obtain an element g of H_n such that $g(x) \geqslant 0$ for each $x \in X$. For each $n = 1, 2,\ldots,$ let $U_n = \{x: d(x, p) < 1/n\}$ and let $y \in X \setminus U_1$. It may be that $X \setminus U_1 = \varnothing$, but for some i, $X/U_i \neq \varnothing$. Without loss of generality we assume $i = 1$. Since p is not an isolated point of X, we may choose $n - 1$ distinct points $x_1^k,\ldots, x_{n-1}^k$ of U_k. Since H_n is a Haar space, there exists $g_k \in H_n$ such that

$$g_k(y) > 0,$$

$$\| g_k \| = 1,$$

and

$$g_k(x_i^k) = 0, \quad i = 1, \cdots, n-1 \qquad [3, \text{p. } 20].$$

In the case $n = 1$, there are no such x_i's.

Consider a second norm on H_n:

$$\| \alpha_1 \phi_1 + \cdots + \alpha_n \phi_n \|_m = \max_i | \alpha_i |.$$

Using the result that any two norms on a finite dimensional space are equivalent, we obtain that if

$$g_k(x) = \alpha_{k1} + \cdots + \alpha_{kn} \phi_n ,$$

then since

$$\| g_k \| = 1 \qquad \forall k,$$

we have

$$\| g_k \|_m \leqslant B, \qquad \forall k,$$

B being a constant. This implies that the coefficients

$$\alpha_{ki} , \qquad k = 1, 2,\ldots, \qquad i = 1,\ldots, n,$$

are uniformly bounded. We may, therefore, choose a subsequence of $\{\alpha_{k1}\}$ which converges to same α_1. We then select a subsequence of the subsequence above so that the corresponding subsequence of α_{k2} converges to some α_2. Continuing in this way we produce a convergent subsequence of g_k, hereafter called g_k, which converges to some g in $\| \ \|$. We observe that since

$$g_k(y) > 0 \qquad \forall k$$

and

$$g(y) = \lim_k g_k(y),$$

it follows that

$$g(y) \geqslant 0.$$

We also observe that since

$$\big| \| g \| - \| g_k \| \big| \leqslant \| g - g_k \|,$$

it must be the case that

$$\| g \| = 1.$$

Let $y' \in X - \{p, y\}$. There exists an arc A from y to y' missing p. There exists an $M > 0$ such that $U_k \cap A = \varnothing$ for $k > M$. Now $g_k(y) > 0$ and g_k has no zeros in $X \backslash U_k$. Hence $g_k(x) > 0$ for each $x \in A$ $k > M$. It follows that $g(y') \geqslant 0$. Since y' was an arbitrary point in X distinct from p, $g(x) \geqslant 0$ for each $x \in X$, $x \neq p$. But p is not isolated; hence, by continuity, $g(x) \geqslant 0$ $\forall x \in X$.

If H_n does not contain a positive function, then for each η, $0 < \eta < 1$, ηg is a best approximation from H_n to the function $f(x) = 1$. This contradicts the Haar uniqueness theorem [3, pp. 22]. Thus H_n contains a positive function. We remark that if $X = [a, b]$, then we can choose p to be a or b. In our example earlier, no point p exists such that $X \backslash \{p\}$ is arcwise connected.

References

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